61th International Mathematical Olympiad

Day 1. Official Solutions

Problem 1. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

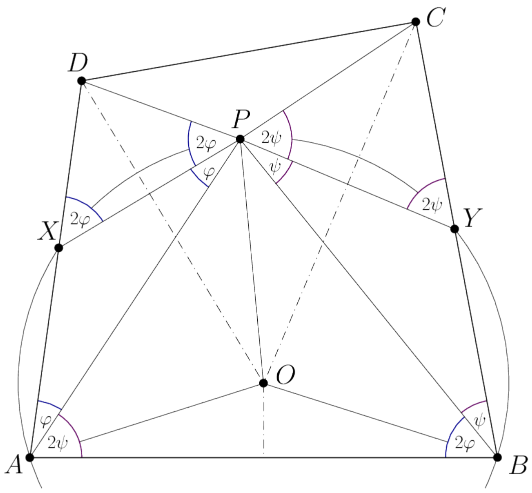
$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Solution 1. Let $\varphi = \angle PAD$ and $\psi = \angle CBP$; then we have $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$. Let X be the point on segment AD with $\angle XPA = \varphi$. Then

$$\angle PXD = \angle PAX + \angle XPA = 2\varphi = \angle DPA - \angle XPA = \angle DPX.$$

It follows that triangle DPX is isosceles with $DX = DP$ and therefore the internal angle bisector of $\angle ADP$ coincides with the perpendicular bisector of XP. Similarly, if Y is a point on BC such that $\angle BPY = \psi$, then the internal angle bisector of $\angle PCB$ coincides with the perpendicular bisector of PY. Hence, we have to prove that the perpendicular bisectors of XP, PY, and AB are concurrent.



Notice that

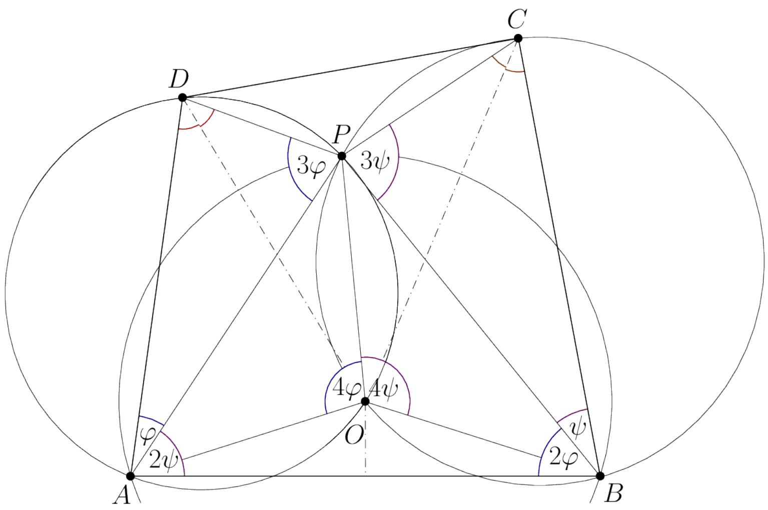
$$\angle AXP = 180^\circ - \angle PXD = 180^\circ - 2\varphi = 180^\circ - \angle PBA.$$

Hence the quadrilateral $AXPB$ is cyclic; in other words, X lies on the circumcircle of triangle APB. Similarly, Y lies on the circumcircle of triangle APB. It follows that the perpendicular bisectors of XP, PY, and AB all pass through the center of circle $(ABYPX)$. This finishes the proof.

Comment. Introduction of points X and Y seems to be the key step in the solution above. Note that the same point X could be introduced in different ways, e.g., as the point on the ray CP beyond P such that $\angle PBX = \varphi$, or as a point where the circle (APB) meets again AB. Different definitions of X could lead to different versions of the further solution.

Solution 2. We define the angles $\varphi = \angle PAD$, $\psi = \angle CBP$ and use $\angle PBA = 2\varphi$, $\angle DPA = 3\varphi$, $\angle BAP = 2\psi$ and $\angle BPC = 3\psi$ again. Let O be the circumcenter of $\triangle APB$.

Notice that $\angle ADP = 180° - \angle PAD - \angle DPA = 180° - 4\varphi$, which, in particular, means that $4\varphi < 180°$. Further, $\angle POA = 2\angle PBA = 4\varphi = 180° - \angle ADP$, therefore the quadrilateral $ADPO$ is cyclic. By $AO = OP$, it follows that $\angle ADO = \angle ODP$. Thus DO is the internal bisector of $\angle ADP$. Similarly, CO is the internal bisector of $\angle PCB$.



Finally, O lies on the perpendicular bisector of AB as it is the circumcenter of $\triangle APB$. Therefore the three given lines in the problem statement concur at point O.

Problem 2. The real numbers a, b, c, d are such that $a \geqslant b \geqslant c \geqslant d > 0$ and $a+b+c+d=1$. Prove that

$$(a+2b+3c+4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

Solution 1. The weighted AM–GM inequality with weights a, b, c, d gives

$$a^a b^b c^c d^d \leqslant a \cdot a + b \cdot b + c \cdot c + d \cdot d = a^2 + b^2 + c^2 + d^2,$$

so it suffices to prove that $(a+2b+3c+4d)(a^2+b^2+c^2+d^2) < 1 = (a+b+c+d)^3$. This can be done in various ways, for example:

$$\begin{aligned}(a+b+c+d)^3 &> a^2(a+3b+3c+3d) + b^2(3a+b+3c+3d)\\ &\quad + c^2(3a+3b+c+3d) + d^2(3a+3b+3c+d)\\ &\geqslant (a^2+b^2+c^2+d^2)\cdot(a+2b+3c+4d).\end{aligned}$$

Solution 2. From $b \geqslant d$ we get

$$a+2b+3c+4d \leqslant a+3b+3c+3d = 3-2a.$$

If $a < \frac{1}{2}$, then the statement can be proved by

$$(a+2b+3c+4d)\, a^a b^b c^c d^d \leqslant (3-2a)a^a a^b a^c a^d = (3-2a)a = 1-(1-a)(1-2a) < 1.$$

From now on we assume $\frac{1}{2} \leqslant a < 1$.

By $b, c, d < 1-a$ we have

$$b^b c^c d^d < (1-a)^b \cdot (1-a)^c \cdot (1-a)^d = (1-a)^{1-a}.$$

Therefore,

$$(a+2b+3c+4d) a^a b^b c^c d^d < (3-2a)\, a^a\, (1-a)^{1-a}.$$

For $0 < x < 1$, consider the functions

$$f(x) = (3-2x)x^x(1-x)^{1-x} \quad \text{and} \quad g(x) = \log f(x) = \log(3-2x) + x\log x + (1-x)\log(1-x);$$

hereafter, log denotes the natural logarithm. It is easy to verify that

$$g''(x) = -\frac{4}{(3-2x)^2} + \frac{1}{x} + \frac{1}{1-x} = \frac{1+8(1-x)^2}{x(1-x)(3-2x)^2} > 0,$$

so g is strictly convex on $(0,1)$.

By $g\left(\frac{1}{2}\right) = \log 2 + 2\cdot\frac{1}{2}\log\frac{1}{2} = 0$ and $\lim\limits_{x\to 1-} g(x) = 0$, we have $g(x) \leqslant 0$ (and hence $f(x) \leqslant 1$) for all $x \in \left[\frac{1}{2}, 1\right)$, and therefore

$$(a+2b+3c+4d) a^a b^b c^c d^d < f(a) \leqslant 1.$$

Comment. For a large number of variables $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ with $\sum_i a_i = 1$, the inequality

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} \leqslant 1$$

does not necessarily hold. Indeed, let $a_2 = a_3 = \ldots = a_n = \varepsilon$ and $a_1 = 1-(n-1)\varepsilon$, where n and $\varepsilon \in (0, 1/n)$ will be chosen later. Then

$$\left(\sum_i i a_i\right) \prod_i a_i^{a_i} = \left(1 + \frac{n(n-1)}{2}\varepsilon\right) \varepsilon^{(n-1)\varepsilon}(1-(n-1)\varepsilon)^{1-(n-1)\varepsilon}. \tag{1}$$

If $\varepsilon = C/n^2$ with an arbitrary fixed $C > 0$ and $n \to \infty$, then the factors $\varepsilon^{(n-1)\varepsilon} = \exp((n-1)\varepsilon\log\varepsilon)$ and $(1-(n-1)\varepsilon)^{1-(n-1)\varepsilon}$ tend to 1, so the limit of (1) in this set-up equals $1 + C/2$. This is not simply greater than 1, but it can be arbitrarily large.

Problem 3. There are $4n$ pebbles of weights $1, 2, 3, \dots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each colour.

Solution 1. Let us pair the pebbles with weights summing up to $4n+1$, resulting in the set S of $2n$ pairs: $\{1, 4n\}$, $\{2, 4n-1\}$, ..., $\{2n, 2n+1\}$. It suffices to partition S into two sets, each consisting of n pairs, such that each set contains two pebbles of each color.

Introduce a multi-graph G (i.e., a graph with loops and multiple edges allowed) on n vertices, so that each vertex corresponds to a color. For each pair of pebbles from S, we add an edge between the vertices corresponding to the colors of those pebbles. Note that each vertex has degree 4. Also, a desired partition of the pebbles corresponds to a coloring of the edges of G in two colors, say red and blue, so that each vertex has degree 2 with respect to each color (i.e., each vertex has equal red and blue degrees).

To complete the solution, it suffices to provide such a coloring for each component G' of G. Since all degrees of the vertices are even, in G' there exists an Euler circuit C (i.e., a circuit passing through each edge of G' exactly once). Note that the number of edges in C is even (it equals twice the number of vertices in G'). Hence all the edges can be colored red and blue so that any two edges adjacent in C have different colors (one may move along C and color the edges one by one alternating red and blue colors). Thus in G' each vertex has equal red and blue degrees, as desired.

Comment 1. To complete Solution 1, any partition of the edges of G into circuits of even lengths could be used. In the solution above it was done by the reference to the well-known Euler Circuit Lemma: Let G be a connected graph with all its vertices of even degrees. Then there exists a circuit passing through each edge of G exactly once.

Solution 2. As in Solution 1, we will show that it is possible to partition $2n$ pairs $\{1, 4n\}$, $\{2, 4n-1\}$, ..., $\{2n, 2n+1\}$ into two sets, each consisting of n pairs, such that each set contains two pebbles of each color.

Introduce a multi-graph (i.e., a graph with multiple edges allowed) Γ whose vertices correspond to pebbles; thus we have $4n$ vertices of n colors so that there are four vertices of each color. Connect pairs of vertices $\{1, 4n\}$, $\{2, 4n-1\}$, ..., $\{2n, 2n+1\}$ by $2n$ *black* edges.

Further, for each monochromatic quadruple of vertices i, j, k, ℓ we add a pair of *grey* edges forming a matching, e.g., (i, j) and (k, ℓ). In each of n colors of pebbles we can choose one of three possible matchings; this results in 3^n ways of constructing grey edges. Let us call each of 3^n possible graphs Γ a *cyclic* graph. Note that in a cyclic graph Γ each vertex has both black and grey degrees equal to 1. Hence Γ is a union of disjoint cycles, and in each cycle black and grey edges alternate (in particular, all cycles have even lengths).

It suffices to find a cyclic graph with all its cycle lengths divisible by 4. Indeed, in this case, for each cycle we start from some vertex, move along the cycle and recolor the black edges either to red or to blue, alternating red and blue colors. Now blue and red edges define the required partition, since for each monochromatic quadruple of vertices the grey edges provide a bijection between the endpoints of red and blue edges.

Among all possible cyclic graphs, let us choose graph Γ_0 having the minimal number of components (i.e., cycles). The following claim completes the solution.

Claim. In Γ_0, all cycle lengths are divisible by 4.

Proof. Assuming the contrary, choose a cycle C_1 with an odd number of grey edges. For some color c the cycle C_1 contains exactly one grey edge joining two vertices i, j of color c, while the other edge joining two vertices k, ℓ of color c lies in another cycle C_2. Now delete edges (i, j) and (k, ℓ) and add edges (i, k) and (j, ℓ). By this switch we again obtain a cyclic graph Γ_0' and decrease the number of cycles by 1. This contradicts the choice of Γ_0. □

Comment 2. Use of an auxiliary graph and reduction to a new problem in terms of this graph is one of the crucial steps in both solutions presented. In fact, graph G from Solution 1 could be obtained from any graph Γ from Solution 2 by merging the vertices of the same color.